1 Exercices

Exercice 1 : Résoudre les équations suivantes, d'inconnue $(x, y) \in \mathbb{R}^2$.

1. $x(1,3) + y(-2,5) = 0_{\mathbb{R}^2}$
2. $x(1,3) + y(-2,5) = (3,3)$
3. $x(2,4) + y(6,12) = 0_{\mathbb{R}^2}$

Exercice 2 : Résoudre les équations suivantes, d'inconnue $(x, y, z) \in \mathbb{R}^3$.

1. $x\begin{pmatrix}1\\1\\1\end{pmatrix} + y\begin{pmatrix}2\\3\\0\end{pmatrix} + z\begin{pmatrix}-1\\0\\4\end{pmatrix} = \begin{pmatrix}0\\-2\\3\end{pmatrix}$
2. à venir...

Exercice 3 :(Commutant) Résoudre les équations suivantes, d'inconnue $M \in \mathscr{M}_2(\mathbb{R})$.

1. $AM = MA$ où $A = \begin{pmatrix}2 & 0\\0 & 0\end{pmatrix}$
2. $AM = MA$ où $A = \begin{pmatrix}2 & 0\\0 & 3\end{pmatrix}$
3. $AM = MA$ où $A = \begin{pmatrix}2 & -1\\-2 & 3\end{pmatrix}$

Exercice 4 : Résoudre les équations suivantes, d'inconnue $P(X) \in \mathbb{R}_2[X]$.

1. $P(X) = P(-X)$
2. $P(X) = -P(-X)$
3. $XP'(X) = 4P(X)$
4. $P(X+1) - P(X-1) = 0_{\mathbb{R}_2[X]}$
5. $P(X) + P'(X) = 0_{\mathbb{R}_2[X]}$

2 Réponses courtes

Réponses de l'exercice 1 : On note S l'ensemble des solutions.

1. $S = \{(0,0)\} = \{0_{\mathbb{R}^2}\}$
2. $S = \left\{\left(\frac{21}{11}, -\frac{6}{11}\right)\right\}$
3. $S = \text{Vect}\left((-3,1)\right)$

Réponses de l'exercice 2 : On note S l'ensemble des solutions.

1. $S = \left\{\left(\frac{19}{5}, -2, -\frac{1}{5}\right)\right\}$

Réponses de l'exercice 3 : On note S l'ensemble des solutions.

1. $S = \text{Vect}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$
2. $S = \text{Vect}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$
3. $S = \text{Vect}\left(\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$

Réponses de l'exercice 4 : On note S l'ensemble des solutions.

1. $S = \text{Vect}\left(X^2, 1\right)$
2. $S = \text{Vect}\left(X\right)$
3. $S = \left\{0_{\mathbb{R}_2[X]}\right\}$
4. $S = \text{Vect}\left(1\right)$
5. $S = \left\{0_{\mathbb{R}_2[X]}\right\}$

3 Corrections détaillées

Correction détaillée de l'exercice 1 : On note S l'ensemble des solutions.

1. Soit $(x,y) \in \mathbb{R}^2$.

$$\begin{aligned}
x(1,3)+y(-2,5)=0_{\mathbb{R}^2} &\iff (x-2y,3x+5y)=0_{\mathbb{R}^2}\\
&\iff \left\{\begin{array}{rcrcl} x & - & 2y & = & 0\\ 3x & + & 5y & = & 0\end{array}\right.\\
&\iff \left\{\begin{array}{rcrcll} x & - & 2y & = & 0 & \\ & & 11y & = & 0 & L_2 \leftarrow L_2-3L_1\end{array}\right.\\
&\iff \left\{\begin{array}{rcrcll} 11x & & & = & 0 & L_1 \leftarrow 11L_1+2L_2\\ & & 11y & = & 0 & \end{array}\right.\\
&\iff x=y=0
\end{aligned}$$

D'où $S=\{(0,0)\}=\{0_{\mathbb{R}^2}\}$.

2. Soit $(x,y) \in \mathbb{R}^2$.

$$\begin{aligned}
x(1,3)+y(-2,5)=(3,3) &\iff (x-2y,3x+5y)=(3,3)\\
&\iff \left\{\begin{array}{rcrcl} x & - & 2y & = & 3\\ 3x & + & 5y & = & 3\end{array}\right.\\
&\iff \left\{\begin{array}{rcrcll} x & - & 2y & = & 3 & \\ & & 11y & = & -6 & L_2 \leftarrow L_2-3L_1\end{array}\right.\\
&\iff \left\{\begin{array}{rcrcll} 11x & & & = & 21 & L_1 \leftarrow 11L_1+2L_2\\ & & 11y & = & -6 & \end{array}\right.\\
&\iff \left\{\begin{array}{rcrcl} x & & & = & \frac{21}{11}\\ & & y & = & -\frac{6}{11}\end{array}\right.
\end{aligned}$$

D'où $S=\left\{\left(\frac{21}{11},-\frac{6}{11}\right)\right\}$.

3. Soit $(x,y) \in \mathbb{R}^2$.

$$\begin{aligned}
x(2,4)+y(6,12)=0_{\mathbb{R}^2} &\iff (2x+6y,4x+12y)=0_{\mathbb{R}^2}\\
&\iff \left\{\begin{array}{rcrcl} 2x & + & 6y & = & 0\\ 4x & + & 12y & = & 0\end{array}\right.\\
&\iff \left\{\begin{array}{rcrcll} x & + & 3y & = & 0 & L_1 \leftarrow \frac{1}{2}L_1\\ x & + & 3y & = & 0 & L_2 \leftarrow \frac{1}{4}L_2\end{array}\right.\\
&\iff x+3y=0\\
&\iff x=-3y
\end{aligned}$$

D'où

$$\begin{aligned}
S &= \left\{(x,y)\in\mathbb{R}^2 \mid x=-3y\right\}\\
&= \left\{(-3y,y)\in\mathbb{R}^2 \mid y\in\mathbb{R}\right\}\\
&= \left\{y(-3,1)\in\mathbb{R}^2 \mid y\in\mathbb{R}\right\}\\
&= \mathrm{Vect}\left((-3,1)\right)
\end{aligned}$$

Correction détaillée de l'exercice 2 : On note S l'ensemble des solutions.

1. Soit $(x, y, z) \in \mathbb{R}^3$.

$$x\begin{pmatrix}1\\1\\1\end{pmatrix} + y\begin{pmatrix}2\\3\\0\end{pmatrix} + z\begin{pmatrix}-1\\0\\4\end{pmatrix} = \begin{pmatrix}0\\-2\\3\end{pmatrix} \iff \begin{pmatrix}x+2y-z\\x+3y\\x+4z\end{pmatrix} = \begin{pmatrix}0\\-2\\3\end{pmatrix}$$
$$\iff \left\{\begin{array}{ccccccc} x & + & 2y & - & z & = & 0 \\ x & + & 3y & & & = & -2 \\ x & & & + & 4z & = & 3 \end{array}\right.$$
$$\iff \left\{\begin{array}{cccccccl} x & + & 2y & - & z & = & 0 & \\ & & y & & & = & -2 & L_2 \leftarrow L_2 - L_1 \\ & - & 2y & + & 5z & = & 3 & L_3 \leftarrow L_3 - L_1 \end{array}\right.$$
$$\iff \left\{\begin{array}{cccccccl} x & + & 2y & - & z & = & 0 & \\ & & y & & & = & -2 & \\ & & & & 5z & = & -1 & L_3 \leftarrow L_3 + 2L_2 \end{array}\right.$$
$$\iff \left\{\begin{array}{cccccl} 5x & + & 10y & = & -1 & L_1 \leftarrow 5L_1 + L_3 \\ & & y & = & -2 & \\ & & 5z & = & -1 & \end{array}\right.$$
$$\iff \left\{\begin{array}{cccccl} 5x & & & = & 19 & L_1 \leftarrow L_1 - 10L_2 \\ & y & & = & -2 & \\ & & 5z & = & -1 & \end{array}\right.$$
$$\iff \left\{\begin{array}{ccccc} x & & & = & \frac{19}{5} \\ & y & & = & -2 \\ & & z & = & -\frac{1}{5} \end{array}\right.$$

D'où $S = \left\{\left(\frac{19}{5}, -2, -\frac{1}{5}\right)\right\}$.

Correction détaillée de l'exercice 3 : On note S l'ensemble des solutions.

1. Soit $M = \begin{pmatrix}a & b\\c & d\end{pmatrix} \in \mathscr{M}_2(\mathbb{R})$.

$$AM = MA \iff \begin{pmatrix}2 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}a & b\\c & d\end{pmatrix} = \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}2 & 0\\0 & 0\end{pmatrix}$$
$$\iff \begin{pmatrix}2a & 2b\\0 & 0\end{pmatrix} = \begin{pmatrix}2a & 0\\2c & 0\end{pmatrix}$$
$$\iff \begin{pmatrix}0 & 2b\\-2c & 0\end{pmatrix} = 0_{\mathscr{M}_2(\mathbb{R})}$$
$$\iff \left\{\begin{array}{cccc} 2b & & = & 0 \\ & -2c & = & 0 \end{array}\right.$$
$$\iff b = c = 0$$

D'où

$$S = \left\{\begin{pmatrix}a & b\\c & d\end{pmatrix} \in \mathscr{M}_2(\mathbb{R}) \mid b = c = 0\right\}$$
$$= \left\{\begin{pmatrix}a & 0\\0 & d\end{pmatrix} \in \mathscr{M}_2(\mathbb{R}) \mid (a, d) \in \mathbb{R}^2\right\}$$
$$= \left\{a\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + d\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \in \mathscr{M}_2(\mathbb{R}) \mid (a, d) \in \mathbb{R}^2\right\}$$
$$= \text{Vect}\left(\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\right)$$

2. Soit $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathscr{M}_2(\mathbb{R})$.

$$\begin{aligned} AM = MA &\iff \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \\ &\iff \begin{pmatrix} 2a & 2b \\ 3c & 3d \end{pmatrix} = \begin{pmatrix} 2a & 3b \\ 2c & 3d \end{pmatrix} \\ &\iff \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix} = 0_{\mathscr{M}_2(\mathbb{R})} \\ &\iff \left\{ \begin{array}{rcl} -b & = & 0 \\ c & = & 0 \end{array} \right. \\ &\iff b = c = 0 \end{aligned}$$

D'où

$$\begin{aligned} S &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathscr{M}_2(\mathbb{R}) \mid b = c = 0 \right\} \\ &= \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \mathscr{M}_2(\mathbb{R}) \mid (a,d) \in \mathbb{R}^2 \right\} \\ &= \left\{ a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathscr{M}_2(\mathbb{R}) \mid (a,d) \in \mathbb{R}^2 \right\} \\ &= \operatorname{Vect}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \end{aligned}$$

3. Soit $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathscr{M}_2(\mathbb{R})$.

$$\begin{aligned}
AM = MA &\iff \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix} \\
&\iff \begin{pmatrix} 2a-c & 2b-d \\ -2a+3c & -2b+3d \end{pmatrix} = \begin{pmatrix} 2a-2b & -a+3b \\ 2c-2d & -c+3d \end{pmatrix} \\
&\iff \begin{pmatrix} 2b-c & a-b-d \\ -2a+c+2d & -2b+c \end{pmatrix} = 0_{\mathscr{M}_2(\mathbb{R})} \\
&\iff \begin{cases} a - b - d = 0 \\ -2a + c + 2d = 0 \\ 2b - c = 0 \\ -2b + c = 0 \end{cases} \\
&\iff \begin{cases} a - b - d = 0 \\ -2b + c = 0 & L_2 \leftarrow L_2 + 2L_1 \\ 2b - c = 0 \\ -2b + c = 0 \end{cases} \\
&\iff \begin{cases} a - b - d = 0 \\ -2b + c = 0 \\ 0 = 0 & L_3 \leftarrow L_3 + L_2 \\ 0 = 0 & L_4 \leftarrow L_4 - L_2 \end{cases} \\
&\iff \begin{cases} a - b - d = 0 \\ -2b + c = 0 \end{cases} \\
&\iff \begin{cases} a - b = d \\ -2b = -c \end{cases} \qquad \textit{(Il faut 2 variables auxiliaires : on choisit c et d)} \\
&\iff \begin{cases} a - b = d \\ 2b = c \end{cases} \\
&\iff \begin{cases} 2a = 2d + c & L_1 \leftarrow 2L_1 + L_2 \\ 2b = c \end{cases} \\
&\iff \begin{cases} a = d + \frac{1}{2}c \\ b = \frac{1}{2}c \end{cases}
\end{aligned}$$

D'où

$$\begin{aligned}
S &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathscr{M}_2(\mathbb{R}) \mid a = d + \frac{1}{2}c, b = \frac{1}{2}c \right\} \\
&= \left\{ \begin{pmatrix} d + \frac{1}{2}c & \frac{1}{2}c \\ c & d \end{pmatrix} \in \mathscr{M}_2(\mathbb{R}) \mid (c,d) \in \mathbb{R}^2 \right\} \\
&= \left\{ c\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathscr{M}_2(\mathbb{R}) \mid (c,d) \in \mathbb{R}^2 \right\} \\
&= \mathrm{Vect}\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \\
&= \mathrm{Vect}\left(\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)
\end{aligned}$$

(On retrouve nos 2 variables auxiliaires à droite : aucune condition ne porte sur elles, donc elles varient chacune dans $\mathbb{R}$)

(En multipliant la première matrice par 2)

Correction détaillée de l'exercice 4 : On note S l'ensemble des solutions.

1. Soit $P(X) = aX^2 + bX + c \in \mathbb{R}_2[X]$.

$$\begin{aligned}
P(X) = P(-X) &\iff aX^2 + bX + c = aX^2 - bX + c \\
&\iff 2bX = 0_{\mathbb{R}_2[X]} \\
&\iff 2b = 0 \\
&\iff b = 0
\end{aligned}$$

D'où

$$\begin{aligned} S &= \left\{aX^2 + bX + c \in \mathbb{R}_2[X] \mid b = 0\right\} \\ &= \left\{aX^2 + c \in \mathbb{R}_2[X] \mid (a,c) \in \mathbb{R}^2\right\} \\ &= \text{Vect}\left(X^2, 1\right) \end{aligned}$$

2. Soit $P(X) = aX^2 + bX + c \in \mathbb{R}_2[X]$.

$$\begin{aligned} P(X) = -P(-X) &\iff aX^2 + bX + c = -(aX^2 - bX + c) \\ &\iff aX^2 + bX + c = -aX^2 + bX - c \\ &\iff 2aX^2 + 2c = 0_{\mathbb{R}_2[X]} \\ &\iff \left\{\begin{array}{rcl} 2a & = & 0 \\ 2c & = & 0 \end{array}\right. \\ &\iff a = c = 0 \end{aligned}$$

D'où

$$\begin{aligned} S &= \left\{aX^2 + bX + c \in \mathbb{R}_2[X] \mid a = c = 0\right\} \\ &= \{bX \in \mathbb{R}_2[X] \mid b \in \mathbb{R}\} \\ &= \text{Vect}\,(X) \end{aligned}$$

3. Soit $P(X) = aX^2 + bX + c \in \mathbb{R}_2[X]$.

$$\begin{aligned} XP'(X) = 4P(X) &\iff X(2aX + b) = 4(aX^2 + bX + c) \\ &\iff 2aX^2 + bX = 4aX^2 + 4bX + 4c \\ &\iff 2aX^2 + 3bX + 4c = 0_{\mathbb{R}_2[X]} \\ &\iff \left\{\begin{array}{rcl} 2a & = & 0 \\ 3b & = & 0 \\ 4c & = & 0 \end{array}\right. \\ &\iff a = b = c = 0 \end{aligned}$$

D'où $S = \left\{0_{\mathbb{R}_2[X]}\right\}$.

4. Soit $P(X) = aX^2 + bX + c \in \mathbb{R}_2[X]$.

$$\begin{aligned} P(X+1) - P(X-1) = 0_{\mathbb{R}_2[X]} &\iff a(X+1)^2 + b(X+1) + c - \left(a(X-1)^2 + b(X-1) + c\right) = 0_{\mathbb{R}_2[X]} \\ &\iff aX^2 + (2a+b)X + (a+b+c) - \left(aX^2 + (-2a+b)X + (a-b+c)\right) = 0_{\mathbb{R}_2[X]} \\ &\iff 4aX + 2b = 0_{\mathbb{R}_2[X]} \\ &\iff \left\{\begin{array}{rcl} 4a & = & 0 \\ 2b & = & 0 \end{array}\right. \\ &\iff a = b = 0 \end{aligned}$$

D'où

$$\begin{aligned} S &= \left\{aX^2 + bX + c \in \mathbb{R}_2[X] \mid a = b = 0\right\} \\ &= \{c \in \mathbb{R}_2[X] \mid c \in \mathbb{R}\} \\ &= \text{Vect}\,(1) \end{aligned}$$

5. Soit $P(X) = aX^2 + bX + c \in \mathbb{R}_2[X]$.

$$\begin{aligned}
P(X) + P'(X) = 0_{\mathbb{R}_2[X]} &\iff (aX^2 + bX + c) + (2aX + b) = 0_{\mathbb{R}_2[X]} \\
&\iff aX^2 + (2a+b)X + (b+c) = 0_{\mathbb{R}_2[X]} \\
&\iff \begin{cases} a = 0 \\ 2a + b = 0 \\ b + c = 0 \end{cases} \\
&\iff \begin{cases} a = 0 \\ b = 0 & L_2 \leftarrow L_2 - 2L_1 \\ b + c = 0 \end{cases} \\
&\iff \begin{cases} a = 0 \\ b = 0 \\ c = 0 & L_3 \leftarrow L_3 - L_2 \end{cases} \\
&\iff a = b = c = 0
\end{aligned}$$

D'où $S = \{0_{\mathbb{R}_2[X]}\}$.